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## LETTER TO THE EDITOR

**Universal spectral form factor for chaotic dynamics****Stefan Heusler<sup>1</sup>, Sebastian Müller<sup>1</sup>, Petr Braun<sup>1,2</sup> and Fritz Haake<sup>1</sup>**<sup>1</sup> Fachbereich Physik, Universität Duisburg-Essen, 45117 Essen, Germany<sup>2</sup> Institute of Physics, Saint-Petersburg University, St Petersburg 198504, Russia

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Online at [stacks.iop.org/JPhysA/37/L31](http://stacks.iop.org/JPhysA/37/L31) (DOI: 10.1088/0305-4470/37/3/L02)**Abstract**

We consider the semiclassical limit of the spectral form factor  $K(\tau)$  of fully chaotic dynamics. Starting from the Gutzwiller-type double sum over classical periodic orbits we set out to recover the universal behaviour predicted by random-matrix theory, both for dynamics with and without time reversal invariance. For times smaller than half the Heisenberg time  $T_H \propto \hbar^{-f+1}$ , we extend the previously known  $\tau$ -expansion to include the cubic term. Beyond confirming the random-matrix behaviour of individual spectra, the virtue of that extension is that the ‘diagrammatic rules’ come in sight which determine the families of orbit pairs responsible for all orders of the  $\tau$ -expansion.

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**1. Introduction**

One of the fascinating quantum signatures of chaos is the universal behaviour of the correlation functions of the spectral density of energy levels, for general hyperbolic dynamics [1]. Three universality classes were suggested by Dyson and Wigner; one, called ‘unitary’, has no time reversal symmetry, while the other two do have Hamiltonians  $H$  commuting with an anti-unitary time reversal operator  $\mathcal{T}$ ; if  $\mathcal{T}^2 = 1$  one speaks of the ‘orthogonal’ class while the ‘symplectic’ case has  $\mathcal{T}^2 = -1$ . The Fourier transform of the two-point correlator of the level density, called the spectral form factor, is predicted by random-matrix theory (RMT) [2] as

$$K_{\text{uni}}(\tau) = \tau \quad K_{\text{orth}}(\tau) = 2\tau - \tau \ln(1 + 2\tau) \quad (1)$$

in the unitary and the orthogonal case; here  $\tau$  is a time made dimensionless by referral to the Heisenberg time  $T_H \propto \hbar^{-f+1}$ , with  $f$  being the number of freedoms; the results (1) hold in the (semiclassical) limit of large dimension of the matrix representation of  $H$  and for times up to the Heisenberg time,  $0 \leq \tau \leq 1$ . The orthogonal form factor allows for the Taylor expansion  $K_{\text{orth}}(\tau) = 2\tau - 2\tau^2 + 2\tau^3 + \dots$ , for  $0 \leq \tau < \frac{1}{2}$ .

Understanding the observed fidelity of individual dynamics to RMT has been an elusive goal, in spite of considerable efforts based on parametric level dynamics, semiclassical

periodic-orbit theory, and the so-called nonlinear sigma model [2]. We shall here take a non-trivial step towards that goal, on semiclassical ground.

Periodic-orbit theory in the style of Gutzwiller [2, 3] gives the form factor of an individual spectrum as the double sum

$$K_{\text{po}}(\tau) = T_H^{-2} \left\langle \sum_{\gamma, \gamma'} A_\gamma A_{\gamma'}^* \exp(i(S_\gamma - S_{\gamma'})/\hbar) \delta \left( \tau - \frac{T_\gamma + T_{\gamma'}}{2T_H} \right) \right\rangle \quad (2)$$

where  $S_\gamma$ ,  $T_\gamma$  and  $A_\gamma$  are the action (including the Maslov phase), period and (dimensionless) stability amplitude of the  $\gamma$ th orbit; the angular brackets demand averages (i) over the centre energy  $(E + E')/2$  in the product  $\rho(E)\rho(E')$  of two level densities before doing the Fourier transform w.r.t. the energy difference  $E - E'$  and (ii) over a time interval small compared to the Heisenberg time. Most orbit pairs  $\gamma, \gamma'$  interfere destructively in the double sum (2). Finite contributions arise only from families of pairs wherein the action difference  $S_\gamma - S_{\gamma'}$  can be continuously steered through the quantum scale  $\hbar$  towards zero, by varying parameters defining the family (for early premonitions see [4]). Periods and stability amplitudes do not differ noticeably within such an orbit pair.

Berry's 'diagonal approximation' [5] includes the trivial pairs  $\{\gamma, \gamma\}$  and, given  $\mathcal{T}$  invariance,  $\{\gamma, \bar{\gamma}\}$  where the overbar indicates time reversal; it yields  $K^{(1)} = \beta \sum_\gamma |A_\gamma|^2 \delta(\tau - \frac{T_\gamma}{T_H}) = \beta\tau$  where  $\beta = 1$  without and  $\beta = 2$  with  $\mathcal{T}$  invariance, due to the doubling of contributing pairs in the latter case. The sum  $\sum_\gamma |A_\gamma|^2 \delta(\tau - \frac{T_\gamma}{T_H}) = \tau$ , known as the sum rule of Hannay and Ozorio de Almeida (HODA) [6], reflects ergodicity for long periodic orbits. In view of (1) the diagonal approximation gives  $K_{\text{uni}}$  in full, and the first term of the  $\tau$ -expansion of  $K_{\text{orth}}$ .

Sieber and Richter [7] recently found a family of orbit pairs which for the  $\mathcal{T}$  invariant Hadamard–Gutzwiller model yields the quadratic term of the  $\tau$ -expansion,  $K_{\text{orth}}^{(2)} = -2\tau^2$ , as in (1). Each Sieber–Richter (SR) pair has a close self-encounter which in configuration space looks like a small-angle crossing for one orbit and like a narrowly avoided crossing for the partner orbit. Generalizations to arbitrary hyperbolic systems with two freedoms were given in [8–10] and for more freedoms in [11].

Before identifying the new families of orbit pairs giving  $K_{\text{orth}}^{(3)} = 2\tau^3$ ,  $K_{\text{uni}}^{(3)} = 0$  we must briefly review SR. Each self-encounter involves two orbit stretches which are nearly mutually time reversed; this may be depicted as  $\rightleftarrows$  (or  $\bowtie$ ), the arrows indicating a sense of traversal. On either side of the 'encounter graph'  $\rightleftarrows$ , each of the two orbits has a long loop attached. Assuming symbolic dynamics available (to uniquely define periodic orbits and even, approximately, short orbit stretches by symbol sequences; our results are valid more generally) we could write  $E$  and  $\bar{E}$  for the two (nearly) mutually time reversed orbit stretches in the encounter region,  $R, L$  for the two long loops, and  $\bar{L}$  for the time reverse of  $L$ ; we may thus write  $ER\bar{E}L$  for an orbit and  $ER\bar{E}\bar{L}$ ,  $LE\bar{R}\bar{E}$  for its SR partners [8, 12]. Note that the orbits in a SR pair traverse one loop in the same sense while the senses of traversal are opposite for the other loop; here,  $\mathcal{T}$  invariance is seen as required for SR pairs to exist.

Following [9, 10, 13] we parametrize an encounter with the help of a surface of section  $\mathcal{P}$  transverse to an orbit, say  $\gamma = ER\bar{E}L$ , somewhere within the encounter;  $\mathcal{P}$  is two dimensional for  $f = 2$ , the case we limit ourselves to. The stretch  $E$  pierces through  $\mathcal{P}$  at a point  $x_a$  which can be made the origin of a coordinate system spanned by tangent vectors  $\hat{e}_s$  and  $\hat{e}_u$  to the stable and unstable manifolds of  $\gamma$  through  $x_a$  in  $\mathcal{P}$ . A second piercing is associated with  $\bar{E}$  and thus opposite in sense; it happens after the traversal of the right loop at a point  $x_b$ . We define a close-encounter region by requiring the unstable and stable components of the difference  $\mathcal{T}x_b - x_a$  to respect a classically small bound  $c$  independent of  $\hbar$ ,

$$\mathcal{T}x_b - x_a = u\hat{e}_u + s\hat{e}_s \quad |u| \leq c \quad |s| \leq c. \quad (3)$$

Moving  $\mathcal{P}$  we leave the encounter after a time  $t_u$  given (asymptotically) by  $|u|e^{\lambda t_u} = c$ . Conversely, we find the start of the encounter going backwards in time by  $t_s$  with  $|s|e^{\lambda t_s} = c$ ; here  $\lambda$  is the Lyapunov exponent of the system. The duration of the encounter is thus

$$t_{\text{enc}} = t_s + t_u = \frac{1}{\lambda} \ln \frac{c^2}{|us|}. \quad (4)$$

Roughly, linearization of the dynamics about any point within the encounter breaks down at either end.

As was shown in [8–10, 13] the partner orbit  $ER\bar{E}\bar{L}$  pierces through  $\mathcal{P}$  first at  $x_a^p = x_a + u\hat{e}_u$  and then at  $x_b^p$  with  $\mathcal{T}x_b^p = x_a + s\hat{e}_s$ . Moreover, the orbits in a SR pair differ in action by the area of the parallelogram spanned by the four points  $x_a, x_a^p, \mathcal{T}x_b, \mathcal{T}x_b^p$  [9, 10],

$$\Delta S = us; \quad (5)$$

that product is canonically invariant and thus independent of the precise location of the surface  $\mathcal{P}$ . (The vectors  $\hat{e}_s, \hat{e}_u$  are pairwise normalized as  $\hat{e}_s \wedge \hat{e}_u = 1$ .) Inasmuch as weighty contributions to the form factor must have  $\Delta S = \mathcal{O}(\hbar)$ , we can conclude that the duration of relevant encounters has the order of the Ehrenfest time  $T_E = \frac{1}{\lambda} \ln \frac{c^2}{\hbar}$ , much smaller than the period  $T = \mathcal{O}(T_H)$ .

To evaluate  $K^{(2)}(\tau)$  we need the cumulative duration  $P(u, s|T) du ds$  of all orbit stretches within self-encounters of a long orbit  $\gamma$  of period  $T = \mathcal{O}(T_H)$ , with unstable and stable components of  $\mathcal{T}x_b - x_a$  in  $[u, u + du]$  and  $[s, s + ds]$ . Ergodicity yields ([8–10] use different conventions)

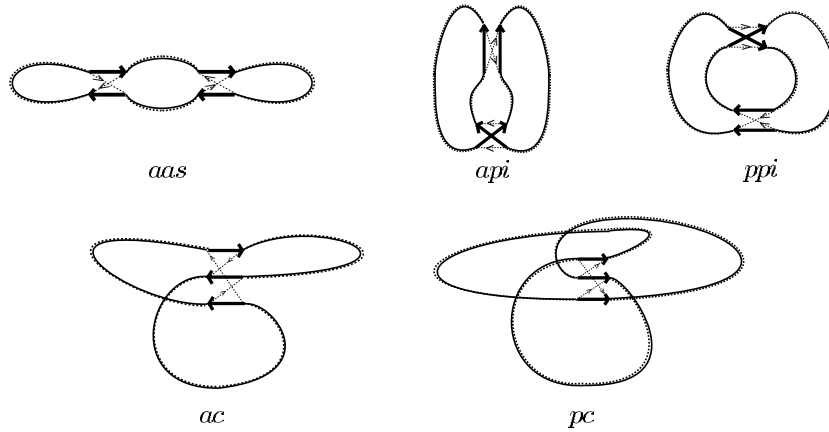
$$P(u, s|T) du ds = T(T - 2t_{\text{enc}})\Omega^{-1} du ds \quad (6)$$

with  $\Omega$  the volume of the energy shell. This results from integrating the ergodic return probability density  $\Omega^{-1}$  over the two times of piercing of the orbit through a section  $\mathcal{P}$ . The factor  $T$  indicates that one piercing, say the one at  $x_a$ , may occur at any time in the interval  $[0, T]$ . The time of the subsequent piercing at  $x_b$  can then lie only in an interval of length  $T - 2t_{\text{enc}}$ , hence the second factor in (6); this is because both traversals of the encounter region have length  $t_{\text{enc}}$  and may not overlap. (Overlapping stretches  $E, \bar{E}$  are either impossible, as in the Hadamard–Gutzwiller model [12], or indicate an orbit with an almost self-retracing loop identical with its SR partner [8].) Note that in the density  $P(u, s|T)$  each encounter is weighted with the duration  $\mathcal{N}t_{\text{enc}}$ ; the combinatorial factor  $\mathcal{N} = 2$  arises since in  $\gamma = ER\bar{E}\bar{L}$  the two stretches  $E, \bar{E}$  are equivalent; we must therefore employ  $\frac{P(u, s|T)}{\mathcal{N}t_{\text{enc}}}$ , to count each encounter only once<sup>3</sup>.

The contribution to the form factor reads  $K_{\text{orth}}^{(2)} = (\sum_{\gamma} |A_{\gamma}|^2 \delta(\tau - \frac{T}{T_H})) \int_{-c}^c du ds \frac{1}{2t_{\text{enc}}} P(u, s|T) 2 \cos(us/\hbar)$ . The sum  $(\sum_{\gamma} \dots)$  gives the factor  $\tau$  through the HODa sum rule, as for  $K^{(1)}$ . The integral gets no contribution from the leading term  $T^2/\Omega$  in  $P$ , and this is why the  $\mathcal{O}(T_E/T_H)$  correction in (6) is important; the integral with  $P \rightarrow -2t_{\text{enc}}T/\Omega$  becomes independent of the bound  $c$  in the limit  $\hbar \rightarrow 0$  and is easily found (since  $t_{\text{enc}}$  cancels from the integrand and  $T_H = \Omega/2\pi\hbar$ ) as  $-2\tau$ . The RMT result  $K^{(2)} = -2\tau^2$  is thus recovered.

One might wonder why no ‘parallel’ encounters with graphs  $\rightleftharpoons$  come into play. The simple reason is that the would-be SR partner of an orbit  $EREL$  decomposes into a ‘pseudo-orbit’ (separate periodic orbits  $ER$  and  $EL$ ), not admitted to the Gutzwiller sum in the first

<sup>3</sup> The appearance of  $P(u, s|T)/2t_{\text{enc}}$  may also be seen as due to the identity  $\int_{-c}^c du ds \delta(us - \Delta S) \frac{1}{2t_{\text{enc}}} P(u, s|T) = \lambda P|_{us=\Delta S}$ ; note that  $u, s$  enter  $P(u, s|T)$  and  $t_{\text{enc}}$  only through the product  $us = \Delta S$ . We may interpret  $\lambda P|_{us=\Delta S} d\Delta S$  as the number of encounters (thus the number of SR partners) with action difference in  $[\Delta S, \Delta S + d\Delta S]$ , for period- $T$  orbits. Analogous identities hold for the densities met with below for  $\tau^3$ .



**Figure 1.** The five orbit pairs entering  $\tau^3$ . Labels describe encounters as antiparallel or antiunitary-symmetry-required, parallel, intertwined, serial, and cloverleaf.

place. However, parallel encounters will be met with below. The quantification (3) of ‘close’ must then be changed to  $x_b - x_a = u\hat{e}_u + s\hat{e}_s$ ,  $|u| \leq c$ ,  $|s| \leq c$ .

The foregoing review of SR has highlighted those twists of the original formulation of [7], in particular the appearance of  $\frac{P(u,s|T)}{2t_{\text{enc}}}$ , which make the extension to higher orders in  $\tau$  a rather elegant travail, to which we now turn.

## 2. Orbit pairs contributing in third order

We now present five families of orbit pairs relevant for  $\tau^3$ . They can be constructed from two SR ‘switches’  $\rightleftharpoons / \bowtie$ . Three families have two separate encounters wherein the orbits differ and four intervening ‘loops’ near identical for both orbits; two more families arise as one of the loops shrinks, to let the two encounters overlap, see figure 1.

Starting with two independent encounters, we must obviously check three possibilities: (*aa*) both ‘antiparallel’, pictorially  $\rightleftharpoons \rightleftharpoons$ , (*ap*) antiparallel and parallel, i.e.  $\rightleftharpoons \rightleftarrows$  and (*pp*) both parallel, i.e.  $\rightleftarrows \rightleftarrows$ . For all of them there are two distinct ways of filling in intervening lines, with the two encounters either in series (*s*) or intertwined (*i*). In symbolic notation, those two ways read for case (*aas*)  $E_1 A \bar{E}_1 B E_2 C \bar{E}_2 D$  (in series) and for case (*aai*)  $E_1 A E_2 B \bar{E}_1 C \bar{E}_2 D$  (intertwined); here  $E_1, \bar{E}_1$  ( $E_2, \bar{E}_2$ ) represent the two orbit stretches of the first (second) antiparallel encounter which are nearly mutually time reversed, while the remaining symbols refer to intervening lines; one immediately checks that only the *aas* orbit allows for a partner orbit,  $E_1 \bar{A} \bar{E}_1 B E_2 \bar{C} \bar{E}_2 D$ , while the would-be partner of the *aai* orbit turns out to be a pseudo-orbit decomposing into the two periodic orbits  $E_1 A E_2 \bar{C}$  and  $E_1 \bar{B} \bar{E}_2 D$ . In the same vein we find that for (*ap*) only the *api* orbit and for (*pp*) only the *ppi* orbit have non-decomposing partners. We have thus identified three families of orbit pairs with two far apart self-encounters. Clearly, *ppi* pairs arise even without  $\mathcal{T}$  invariance, while *aas* and *api* pairs require that symmetry.

Now on to the families resulting from shrinking away one of the four loops in the foregoing families. The three remaining loops make for a cloverleaf (*c*) structure; the encounter region accommodates a triple of oriented short ( $\mathcal{O}(T_E)$ ) stretches with encounter graphs  $\rightleftharpoons$  (*pc*) and  $\rightleftarrows$  (*ac*); in symbolic notation, they involve  $E, E, E$  and  $E, \bar{E}, \bar{E}$ , respectively. Schematically, the two types of cloverleaf orbits look like the thick lines in figure 1 *ac*, *pc*. We shall argue that each has a unique partner, shown as dashed lines in figure 1. The family (*pc*) with three

parallel orbit stretches  $\equiv$  in the encounter does not require  $\mathcal{T}$  invariance while the  $\equiv$  family ( $ac$ ) does.

To check the uniqueness of the partner for  $ac$  and  $pc$  we start from the respective thick lines in figure 1. Each candidate for partnership must have its three long loops nearly identical with those of the starting orbit, save possibly for time reversal; those loops are differently connected in the encounter region. A sextet of cloverleaf orbits thus seems to arise, whose encounter graphs are (without arrows, momentarily)  $\equiv \times \times \times \times \times$ . The last three immediately drop from candidacy since they represent SR partners of  $\equiv$  contributing to  $\tau^2$  or entail decomposing pseudo-orbits. Now putting arrows on  $\equiv$ , say for  $ac$  as  $\equiv$ , and hooking on the three loops as in the thick line of figure 1  $ac$ , we find only  $\times$  to lead to a non-decomposing partner, the dashed line in figure 1  $ac$  with the encounter graph  $\times$  (plus, of course its time reverse); the uniqueness of the  $pc$  partner (up to time reversal, if  $\mathcal{T}$  invariance holds) is shown similarly.

### 3. Duration and action differences

For the independent-encounter pairs  $aas$ ,  $api$ ,  $ppi$  the action differences of the two encounters sum up to  $\Delta S = u_1 s_1 + u_2 s_2$ . The triple encounters  $ac$ ,  $pc$  require extra thought.

Beginning with  $ac$  we employ a surface of section  $\mathcal{P}$  through the encounter region and denote the three points of piercing by  $x_{a,b,c}$ . As in (3) we have  $\mathcal{T}x_b - x_a = u_b \hat{e}_u + s_b \hat{e}_s$ ,  $\mathcal{T}x_c - x_a = u_c \hat{e}_u + s_c \hat{e}_s$ ; the bound  $c$  must be respected by all six distances  $|u_b|$ ,  $|s_b|$ ,  $|u_c|$ ,  $|s_c|$ ,  $|u_b - u_c| \equiv |u_{bc}|$ ,  $|s_b - s_c| \equiv s_{bc}$  and the cloverleaf encounter lasts for

$$t_{\text{enc}}^{\text{cl}} = \frac{1}{\lambda} \ln \frac{c^2}{\max\{|u_b|, |u_c|, |u_{bc}|\} \max\{|s_b|, |s_c|, |s_{bc}|\}}. \quad (7)$$

Again, relevant encounters will have durations of the order of the Ehrenfest time  $t_E = \frac{1}{\lambda} \ln \frac{c^2}{h}$ .

Now look at an  $ac$  pair  $\{\gamma, \gamma^p\}$  with encounter graphs  $\equiv$  and  $\times$ . To find the action difference we proceed in two steps. In the first, we employ an auxiliary orbit, the SR partner  $\gamma'$  of  $\gamma$  related to the encounter of the ‘upper two’ stretches, labelled  $a$ ,  $b$ , leaving the ‘lowest’ stretch  $c$  as in  $\gamma$ . For  $\gamma'$  the encounter region is  $\times$ . According to what we have said above about the four points of piercing through the surface of section of an SR encounter the piercings of  $\gamma'$  occur at  $x'_a = x_a + u_b \hat{e}_u$  and  $x'_b$  with  $\mathcal{T}x'_b = x_a + s_b \hat{e}_s$ ; moreover, the action difference within  $\{\gamma, \gamma'\}$  is given by (5) as  $S_{\gamma'} - S_\gamma = u_b s_b$ . In our second step we arrive at the  $ac$  partner  $\gamma^p$  of  $\gamma$  as the SR partner of  $\gamma'$  w.r.t. the stretches  $a$ ,  $c$ . Once again invoking what we know about the four piercings in an SR encounter we have  $x'_a{}^p = x'_a + (u_c - u_b) \hat{e}_u$  and  $\mathcal{T}x'_c{}^p = x'_a + s_c \hat{e}_s$ , and the action difference  $S_{\gamma^p} - S_{\gamma'} = (u_c - u_b) s_c$ . The action difference of the  $ac$  pair (valid also for  $pc$ ) thus reads

$$\Delta S = S_{\gamma^p} - S_\gamma = u_b s_b + u_c s_c - u_b s_c. \quad (8)$$

The term  $u_b s_c$  represents an ‘interaction’.

### 4. $\tau^3$ -contributions from two simple encounters

We first generalize the density (6) to  $P(u_1, s_1, u_2, s_2|T)$  (up to the factor  $du_1 ds_1 du_2 ds_2$ ) the ‘area’ of times  $(t_a^1, t_a^2) \in [0, T]^2$  such that the points  $\{x_a^\mu = x(t_a^\mu), \mu = 1, 2\}$  of piercing are followed by piercings at  $x_b^\mu = x(t_b^\mu)$  with unstable and stable components of  $x_b^\mu - x_a^\mu$  (for parallel encounters) and  $\mathcal{T}x_b^\mu - x_a^\mu$  (for antiparallel encounters) in the intervals  $[u_\mu, u_\mu + du_\mu]$ ,  $[s_\mu, s_\mu + ds_\mu]$ . We obtain

$$P(u_1, s_1, u_2, s_2|T) = \frac{1}{6\Omega^2} T (T - 2(t_{\text{enc}1} + t_{\text{enc}2}))^3 \quad (9)$$

by integrating the ergodic probability density  $\Omega^{-2}$  for two encounters over the four times  $\{t_a^\mu, t_b^\mu, \mu = 1, 2\}$ , respecting the order of those times dictated by the ‘diagrams’ *aas*, *api*, *ppi* and the general rule that an orbit must leave one encounter region before reentering or entering the next one; the latter rule in fact separates independent-encounter from cloverleaf families. The restrictions on the times of piercing in question give rise to the small but decisive corrections  $t_{\text{enc},\mu}$  in (9), as before in (6).

Again in analogy with (6), the density (9) overcounts a pair of simple encounters, by a factor which we shall argue to be  $\mathcal{N}t_{\text{enc}1}t_{\text{enc}2}$ . Obviously, the times of piercing may lie anywhere during the respective encounters, hence the product of the two durations. The factor  $\mathcal{N}$  is of combinatorial nature. For instance, in *aas* pairs the two antiparallel encounters are indistinguishable such that  $\mathcal{N}_{aas} = 2$ ; likewise,  $\mathcal{N}_{api} = 2$  since the two stretches of, say, the parallel encounter are indistinguishable; finally, in *ppi* pairs all four orbit stretches (symbolically,  $E_1, E_2, E_1, E_2$ ) are indistinguishable, hence  $\mathcal{N}_{ppi} = 4$ .

For  $\mathcal{T}$  invariance, (2) gives  $K_{\text{orth,ind}}^{(3)} = \tau \int_{-c}^c du_1 \dots ds_2 \frac{1}{\mathcal{N}t_{\text{enc}1}t_{\text{enc}2}} P(u_1 \dots s_2|T) 2 \cos \frac{\Delta S}{\hbar}$  where we have already allowed the HOdA sum rule to yield a factor  $\tau$  as above and used the overcounting factor  $\mathcal{N}$  as an indicator for the cases *aas*, *api*, *ppi*. Using the action difference (5) for both encounters and the weight (9) it is easily found that only the part  $\frac{1}{6\Omega^2} \times 3 \times 4 \times T^2 t_{\text{enc}1} t_{\text{enc}2}$  in  $P$  survives the integration, and actually yields the square of the twofold integral met with for  $\tau^2$ ; all other terms and all  $c$  dependence vanish with  $\hbar \rightarrow 0$ . We thus have

$$K_{\text{orth,ind}}^{(3)} = \frac{8}{\mathcal{N}} \tau^3 = \begin{cases} 4\tau^3 & \text{for } aas, \quad api \\ 2\tau^3 & \text{for } ppi. \end{cases} \quad (10)$$

For dynamics without  $\mathcal{T}$  invariance, however, only *ppi* pairs exist and yield  $K_{\text{uni,ind}}^{(3)} = \tau^3$ , one half the *ppi* term in (10) since a *ppi* orbit now has no time reverse.

### 5. $\tau^3$ -contributions from triple encounters

Both for *ac* and *pc* pairs, by reasoning as before we have the density

$$P^{\text{cl}}(u_b, s_b, u_c, s_c|T) = \frac{T(T - 3t_{\text{enc}}^{\text{cl}} + \dots)^2}{2\Omega^2} = -\frac{3T^2 t_{\text{enc}}^{\text{cl}}}{\Omega^2} \dots \quad (11)$$

where the dots point to terms killed by the integration to come, as  $\hbar \rightarrow 0$ . The crucial term  $\propto t_{\text{enc}}^{\text{cl}}$  is due to minimal loop lengths. The orbit must leave an encounter before reentering in the antiparallel or parallel sense; otherwise, an SR pair already accounted for in  $\tau^2$  would arise in the antiparallel case, whereas the parallel case would lead to a new family with stretches involved in encounters resembling multiple repetitions of shorter periodic orbits; such families turn out irrelevant for  $\hbar \rightarrow 0$ .

To count each cloverleaf only once we divide out the familiar  $\mathcal{N}t_{\text{enc}}^{\text{cl}}$ . For *pc* encounters we have  $\mathcal{N}_{pc} = 3$  since the three parallel stretches  $\overleftrightarrow{\Xi}$  are indistinguishable while *ac* encounters  $\overleftrightarrow{\Xi}$  entail  $\mathcal{N}_{ac} = 1$ . Given  $\mathcal{T}$  invariance, the form factor picks up  $K_{\text{orth,cl}}^{(3)} = \tau \int_{-c}^c du_b \dots ds_c \frac{1}{\mathcal{N}t_{\text{enc}}^{\text{cl}}} P^{\text{cl}}(u_b \dots s_c|T) 2 \cos \frac{\Delta S}{\hbar}$ . The limit  $\hbar \rightarrow 0$  yields

$$K_{\text{orth,cl}}^{(3)} = -\frac{6}{\mathcal{N}} \tau^3 = \begin{cases} -6\tau^3 & \text{for } ac \\ -2\tau^3 & \text{for } pc. \end{cases} \quad (12)$$

Without  $\mathcal{T}$  invariance, no *ac* pairs exist and *pc* pairs, not accompanied by time inverses, give but  $K_{\text{uni,cl}}^{(3)} = -\tau^3$ .

## 6. Conclusions and outlook

Adding contributions from independent and cloverleaf encounters we get  $K_{\text{orth}}^{(3)} = (4 + 4 + 2 - 6 - 2)\tau^3 = 2\tau^3$  and  $K_{\text{uni}}^{(3)} = 0$ , as in (1). To third order in  $\tau$  at least, then, semiclassical treatment of *individual* hyperbolic dynamics gives the universal form factor characteristic of *ensembles of random matrices*.

The five families of orbit pairs met here resemble diagrams known from field theoretic treatments of disordered systems [14] and from quantum graphs [15]. The analogy between classical orbits and diagrams in field theory should persist in higher orders. Orbit pairs (alias diagrams) with  $n - 1$  separate simple encounters contribute to  $\tau^n$ ; upon shrinking intervening loops we expect to find all other relevant orbit pairs. The weight of each family includes a correction, due to the ban of encounter overlap and small as a power of  $\frac{T_E}{T_H}$ , exclusively affecting the form factor.

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